

# Geometric Theory of Parshin's Residues. Coboundary Operators for Stratified Spaces and the Reciprocity Law for Residues.

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## Abstract

The article consist of two main parts: an analog of the Leray Theory for Singular Varieties and its application to the Theory of Parshin's Residues. The first part is independent from the second. It uses the theory of Whitney stratifications. The second part is an application of the first. In particular, a geometric and very transparent proof of the Parshin's Reciprocity Law for residues is given.

## Introduction.

In [P1] and [P2] Parshin introduced his notion of the residue of a meromorphic  $n$ -form  $\omega$  on a singular  $n$ -dimensional variety  $V$ . This residue is computed not in a point of  $V$ , but at a complete flag of irreducible subvarieties  $V_n \supset \cdots \supset V_0$ ,  $\dim V_k = k$ . In fact, this residue is the sum of residues, which correspond to the choices of "local irreducible components" of the flag. We prefer to deal with these summands separately. Parshin proved the following Reciprocity Law: Fix a meromorphic  $n$ -form  $\omega$  on  $V$ . Consider all possible flags, which have the same components in all dimensions but  $k$ ,  $0 < k < n$ . Then only finitely many of them give non-zero residues for  $\omega$  and the sum of these residues is zero.

Parshin's constructions are completely algebraic. They work over fields of any characteristic. In the case of complex numbers one can expect a geometric counterpart of this theory. J.-L. Brylinski and D.A. McLaughlin in [BM] introduced a homology class associated to the flag, such that the residue is the integral over this class (they introduced flag-localized homologies, to be able to integrate meromorphic forms). But their construction is not very explicit.

In the second part of this paper we give a geometric description of the residues, constructing in a very transparent way an  $n$ -dimensional cycle, such that the residue is the integral over this cycle. After that, using the theory of Leray homomorphisms for stratified spaces, we prove the Reciprocity Law. Leray homomorphisms are introduced in the first part of the paper, using the theory of stratified spaces due to J. Mather ([Ma]).

In all the paper the coefficient ring is arbitrary, except for Chapter 1.3, where coefficients are in  $\mathbb{R}$ .

Some theorems in the second part of the paper are given without proves. We plan to provide the proves in the next paper.

# 1 Leray homomorphisms for stratified spaces.

## 1.1 Necessary facts from the theory of Whitney stratifications.

**Definition 1.1.1.** Let  $M$  be a smooth manifold without boundary. Let  $V$  be a locally closed subset of  $M$ . By a *Whitney stratification*  $\mathbf{S}$  of  $V$ , we mean a cover of  $V$  by pairwise disjoint smooth (not necessary closed) submanifolds of  $M$ , called strata, which lie in  $V$ , and satisfy the following conditions:

1) It is *locally finite* - each point of  $V$  has an open neighborhood which intersect only finite number of strata.

2) *Condition of the frontier* - for each stratum  $X \in \mathbf{S}$  its boundary  $(\overline{X} \setminus X) \cap V$  is a union of strata.

3) Each pair  $(X, Y)$  of strata satisfies *Whitney conditions a* and *b*:

**Condition a:** For any  $x \in X$  and any sequence  $\{y_n\} \in Y$ , such that  $y_n \rightarrow x$ , if the sequence of tangent planes  $T_{y_n}Y$  converges to some plane  $\tau \subset T_xM$  (in the appropriate Grassmanian bundle over  $M$ ) then  $T_xX \subset \tau$ .

**Condition b:** For any  $x \in X$ , any sequence  $\{y_n\} \in Y$ , and any sequence  $\{x_n\} \in X$ , such that  $y_n \rightarrow x$  and  $x_n \rightarrow x$ , if the sequence of tangent planes  $T_{y_n}Y$  converges to some plane  $\tau \subset T_xM$ , and the sequence of secants  $\overline{x_n y_n}$  converges to some line  $l$  (in some smooth coordinate system in  $M$ ), then  $l \subset \tau$ .

**Remark.** Actually, condition **b** implies condition **a**, so it is enough to require condition **b**.

One can prove, that if a pair of strata  $(X, Y)$  satisfies condition **b** and  $\overline{Y} \cap X \neq \emptyset$ , then  $\dim X < \dim Y$ .

**Notation.** We say, that  $X < Y$  if  $\overline{Y} \cap X \neq \emptyset$ . One can see that this defines a partial order on the set of strata  $\mathbf{S}$ .

**Example.** Consider the surface in  $\mathbb{C}^3$  given by the equation  $y^2 + x^3 - z^2x^2 = 0$ . The singular locus of the surface coincide with the  $z$ -axis. Thus, the  $z$ -axis and its complement gives a subdivision of the surface in two smooth pieces. Easy to prove that this pair satisfy the condition **a**, but doesn't satisfy condition **b** at the origin. Note, that the small neighborhood of the origin looks very different from the neighborhood of any other point of the  $z$ -axis.

It is easy to improve the subdivision in such a way that it satisfies condition **b**: one only needs to consider the origin as a separate stratum.

Whitney showed that if conditions **a** and **b** are satisfied for the pair  $(X, Y)$ , then  $Y$  "behaves regularly" along  $X$ .

**Theorem 1.1.1.** *Let  $V$  be a complex analytic subset in a smooth manifold  $M$ . Let  $\Sigma$  be a locally finite family of complex analytic subsets in  $V$ . Then there exists a Whitney stratification of the set  $V$  such that each element of  $\Sigma$  is a union of strata and all strata are analytic.*

Detailed review of the theory of Whitney stratifications can be found in [GM].

**Definition 1.1.2.** An abstract stratified set is a triple  $\{V, \mathbf{S}, \mathbf{J}\}$  satisfying the following axioms:

(A1)  $V$  is a Hausdorff, locally compact topological space with a countable basis for its topology.

(A2)  $\mathbf{S}$  is a family of locally closed connected subsets of  $V$ , such that  $V$  is the disjoint union of the members of  $\mathbf{S}$ .

Members of  $\mathbf{S}$  are called strata of  $V$ .

(A3) Each stratum of  $V$  is a topological manifold (in the induced topology), provided with a smoothness structure.

(A4) The family  $\mathbf{S}$  is locally finite.

(A5) The family  $\mathbf{S}$  satisfies the axiom of the frontier: if  $X, Y \in \mathbf{S}$  and  $X \cap \overline{Y} \neq \emptyset$ , then  $X \subset \overline{Y}$ .

If  $X \subset \overline{Y}$  and  $Y \neq X$ , we write  $X < Y$ . Easy to see that this defines a partial order on  $\mathbf{S}$ .

(A6)  $\mathbf{J}$  is a triple  $\{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$ , where for each  $X \in \mathbf{S}$ ,  $U_X$  is an open neighborhood of  $X$  in  $V$ ,  $\pi_X$  is a continuous retraction of  $U_X$  onto  $X$ , and  $\rho_X : U_X \rightarrow [0, \infty)$  is a continuous function.

We call  $U_X$  the tubular neighborhood of  $X$ ,  $\pi_X$  the projection to  $X$  and  $\rho_X$  the tubular function of  $X$ .

(A7)  $X = \{v \in U_X : \rho_X(v) = 0\}$ .

If  $X$  and  $Y$  are any strata, we let  $U_{X,Y} = U_X \cap U_Y$ ,  $\pi_{X,Y} = \pi_X|_{U_{X,Y}}$  and  $\rho_{X,Y} = \rho_X|_{U_{X,Y}}$ .

(A8) For any strata  $X$  and  $Y$  the mapping

$$(\pi_{X,Y}, \rho_{X,Y}) : U_{X,Y} \rightarrow X \times (0, \infty)$$

is a smooth submersion.

(A9) For any strata  $X$ ,  $Y$  and  $Z$  we have

$$\pi_{X,Y} \pi_{Y,Z}(v) = \pi_{X,Z}(v)$$

$$\rho_{X,Y} \pi_{Y,Z}(v) = \rho_{X,Z}(v)$$

whenever both sides of these equations are defined.

**Definition 1.1.3.** We say that two stratified sets  $\{V, \mathbf{S}, \mathbf{J}\}$  and  $\{V', \mathbf{S}', \mathbf{J}'\}$  are equivalent if the following conditions hold:

(a)  $V = V'$ ,  $\mathbf{S} = \mathbf{S}'$ , and for each stratum  $X$  of  $\mathbf{S} = \mathbf{S}'$ , the two smoothness structures on  $X$  given by the two stratifications are the same.

(b) If  $\mathbf{J} = \{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$  and  $\mathbf{J}' = \{\{U'_X\}, \{\pi'_X\}, \{\rho'_X\}\}$ , then for each stratum  $X$ , there exist a neighborhood  $U''_X$  of  $X$  in  $U_X \cap U'_X$  such that  $\rho_X|_{U''_X} = \rho'_X|_{U''_X}$  and  $\pi_X|_{U''_X} = \pi'_X|_{U''_X}$ .

It is easy to prove, that any stratified set is equivalent to one which satisfies the following conditions:

(A10) If  $X, Y$  are strata and  $U_{X,Y} \neq \emptyset$ , then  $X < Y$ .

(A11) If  $X, Y$  are strata and  $U_X \cap U_Y \neq \emptyset$ , then  $X$  and  $Y$  are comparable ( $X < Y$  or  $Y < X$ ).

We'll consider only stratified spaces satisfying conditions A10 and A11.

**Definition 1.1.4.** The Triple  $\mathbf{J} = \{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$  is called *control data*.

**Definition 1.1.5.** Let  $V$  be a subset in a smooth manifold  $M$  with a fixed Whitney stratification  $\mathbf{S}$ . A map  $f : M \rightarrow P$ , where  $P$  is a manifold, is called submersion on  $V$  if its restriction to each stratum  $X \in \mathbf{S}$  is a submersion.

**Definition 1.1.6.** Let  $\{V, \mathbf{S}, \mathbf{J}\}$  be a stratified space. A map  $f : V \rightarrow P$ , where  $P$  is a smooth manifold, is called *controlled submersion* if its restriction to each stratum is a submersion, and  $f \circ \pi_X(y) = f(y)$  for all  $X \in \mathbf{S}$ ,  $y \in U_X$ .

**Theorem 1.1.2.** *Let  $V$  be a subset in a smooth manifold  $M$  with a fixed Whitney stratification  $\mathbf{S}$ . Let  $f : M \rightarrow P$  be a submersion on  $V$ . Then there exist a control data  $\mathbf{C}$ , such that  $\{V, \mathbf{S}, \mathbf{C}\}$  is an abstract stratified space and  $f$  is a controlled submersion. Tubular functions and projections are restrictions of norms and projections in tubular neighborhoods of strata in  $M$  (one has to choose appropriate isomorphisms between tubular neighborhoods and normal bundles, endowed with Euclidian structure).*

**Theorem 1.1.3.** *Let  $P$  be a manifold and  $f : V \rightarrow P$  be a proper, controlled submersion of an abstract stratified space. Then  $f$  is a locally trivial fibration and its restrictions to strata are differentiable fibrations.*

The following Thom's First Isotopy Lemma follows from Theorems 1.1.2 and 1.1.3:

**Theorem 1.1.4.** *(First Isotopy Lemma) Let  $V$  be a subset in a smooth manifold  $M$  with a fixed Whitney stratification  $\mathbf{S}$ . Let  $f : M \rightarrow P$  be a smooth map to a smooth manifold  $P$ . Let  $f|_V$  be a proper submersion. Then  $f|_V$  is a locally trivial fibration.*

Control data was introduced by J.Mather in [Ma]. We need the following result, which easily follows from the Mather's theory of controlled vector fields:

**Theorem 1.1.5.** *Let  $U \subset X$  be an open subset of a stratum  $X \in \mathbf{S}$ , such that the closure  $\overline{U}$  is compact subset in  $X$ . Then there exist an open neighborhood  $W \supset U$  in  $V$ , such that*

- (i)  $W \subset U_X$ ;
- (ii)  $\pi_X(W) = U$  and  $\pi_X|_W$  is a locally trivial fibration.
- (iii) For small enough  $\delta$ ,  $N_{U,\delta} = W \cap \rho_X^{-1}(\delta)$  has structure of a stratified space, given by intersections with strata of the original stratification and restrictions of tubular functions and projections. Moreover,  $\pi_X|_{N_{U,\delta}}$  is a locally trivial fibration over  $U$  with compact fiber. (The last statement follows from the Theorem 1.1.3)
- (iv) There is a homeomorphism  $\phi : W \setminus U \rightarrow N_{U,\delta} \times (0, \Delta)$ , which maps strata to strata and which restrictions to strata are diffeomorphisms. Moreover, the following diagram commutes:

$$\begin{array}{ccc}
& & (0, \Delta) \\
& \nearrow \rho_X & \uparrow \pi_2 \\
W \setminus U & \xrightarrow[\phi]{\sim} & N_{U,\delta} \times (0, \Delta) \\
& \searrow \pi_X & \downarrow \pi_1 \\
& & N_{U,\delta} \\
& & \downarrow \pi_X \\
& & U
\end{array}$$

where  $\pi_1$  and  $\pi_2$  are projections of  $N_{U,\delta} \times (0, \Delta)$  to  $N_{U,\delta}$  and  $(0, \Delta)$  respectively.

**Remark.** Using Theorem 1.1.5 one can shrink the tubular neighborhoods in the following way: Let  $U_1 \subset U_2 \subset \dots \subset U_n \subset \dots \subset X$  be a sequence of open subsets of  $X$ , such that for each  $i$  the closure  $\overline{U_i}$  is a compact subset in  $U_{i+1}$  and  $\bigcup U_i = X$ . Let  $W_i$  be the neighborhood of  $U_i$ , as in the Theorem 1.1.5. Then one can consider the union  $\bigcup W_i$  instead of the neighborhood  $U_X$ . For simplicity we'll always assume that our tubular neighborhoods are of this form.

## 1.2 Homomorphisms and relations.

Let all the strata of a stratified space  $V$  be oriented (in the case of an analytic space it is satisfied automatically).

Let  $X \in \mathbf{S}$  be a stratum. Consider a representative of a homology class  $a \in H_n(X)$  given by map  $\alpha : A \rightarrow X$  of a compact simplicial complex  $A$ . Since  $\alpha(A)$  is compact, there exist an open connected subset  $U \subset X$  such that its closure is a compact subset in  $X$  and  $\alpha(A) \subset U$ . According to the Theorem 1.1.5, there exist the fibration  $p' : N_{U,\delta} \rightarrow U$ . Consider such a stratum  $Y > X$  of  $V$ , that there is no stratum  $Z$  such that  $Y > Z > X$ . Lets denote by  $p$  the restriction of  $p'$  to  $(N_{U,\delta} \cap Y)^0$  where  $(N_{U,\delta} \cap Y)^0$  is a connected component of the intersection  $N_{U,\delta} \cap Y$ . Now  $p$  is a smooth fibration.

Note, that all strata of  $N_{U,\delta}$  are naturally oriented. Namely, we say that the restriction a form  $\omega$  defines the positive orientation of a stratum  $Z \cap N_{U,\delta}$  if the form  $d\rho_{X,Z} \wedge \omega$  defines the positive orientation of  $Z$ .

Using the map  $\alpha : A \rightarrow U$  one can induce the fibration  $P : B \rightarrow A$  with the map  $\beta : B \rightarrow (N_{U,\delta} \cap Y)^0$  which completes the commutative diagram:

$$\begin{array}{ccc}
(N_{U,\delta} \cap Y)^0 & \xleftarrow{\beta} & B \\
\downarrow p & & \downarrow P \\
U & \xleftarrow{\alpha} & A
\end{array}$$

**Lemma 1.2.1.**  *$B$  is compact.*

*Proof:* It's enough to prove that the fiber of  $p$  is compact.  $N_{U,\delta} \cap Y$  is a closed subset in  $N_{U,\delta}$ , because if it isn't closed, then there should be such a stratum  $Z$  that  $Y > Z > X$ . Since  $p'$  is proper,  $p'^{-1}(pt)$  is compact. Thus  $p^{-1}(pt) = p'^{-1}(pt) \cap (N_{U,\delta} \cap Y)$  is compact as a closed subset of a compact set.  $\square$

$B$  is fibered over  $A$  with a compact oriented smooth fiber (the orientation on a fiber of  $p : (N_{U,\delta} \cap Y)^0 \rightarrow U$  is given as follows: restriction of a form  $\omega_F$  defines the positive orientation on the fiber if  $p^*(\omega_X) \wedge \omega_F$  defines the positive orientation of the  $N_{U,\delta} \cap Y$ , where  $\omega_X$  defines positive orientation of  $X$ ). One can triangulate  $B$  in such a way that the preimage of a simplex in  $A$  is a union of simplexes in  $B$ . One can assume that  $\alpha$  is smooth on the interiors of the simplexes of the biggest dimension of  $A$ , then  $P$  is also smooth on the interior of simplexes of biggest dimension. Moreover, one can endow  $B$  with the orientation given by the form  $P^*(\omega_A) \wedge \omega_F$ , where  $\omega_A$  defines the orientation on  $A$  and  $\omega_F$  — orientation on the fiber (we consider everything at a point in the interior of a biggest dimension simplex in  $B$ ). Now  $b$  defines a cycle in  $Y$ .

Lets denote by  $b$  the homology class of the map  $\beta : B \rightarrow Y$  in  $H_k(Y)$ , where  $k = n + \dim Y - \dim X - 1$ .

Easy to see that the homology class  $b$  doesn't depend on the choice of  $\delta$  (involved in the construction of  $N_{U,\delta}$ ).

Recall the following

**Definition 1.2.1.** Two sets  $A$  and  $B$  are called equivalent at  $X$  if there is a small open neighborhood  $U \supset X$ , such that  $A \cap U = B \cap U$ . A class of equivalent sets is called a germ of sets at  $X$ . A germ is called connected if for each its representative  $A$  there exist a small neighborhood  $U \supset X$ , such that  $A \cap U$  is connected. A germ of subsets of  $Y$  at  $X$  is called a connected component of  $Y$  at  $X$  if it is connected at  $X$  and for each its representative  $A$  there exist a small neighborhood  $U \supset X$ , such that  $A \cap U$  is open and closed in  $Y \cap U$ .

Note, that in the case when  $Y$  and  $X$  are strata of a stratified space, the connected components of  $Y$  at  $X$  are easy to understand: the connected components of  $Y \cap U_X$  are in one-to-one correspondence with the connected components of  $Y$  at  $X$ . (See the remark in the end of the Section 1.1.)

**Theorem 1.2.1.** *This construction gives a homomorphism  $\phi_{(X,Y)^0} : H_n(X) \rightarrow H_k(Y)$  for each connected component of  $Y$  at  $X$ . (If a connected component of  $Y$  at  $X$  splits into several connected components of  $N_{U,\delta} \cap Y$ , then one needs to sum up cycles for these components.)*

*Proof:* Obvious from the construction.  $\square$

**Remark.** Let  $U \subset X$  be an open subset. In the same way one can construct the homomorphisms from  $H_n(U)$  to  $H_k(Y)$ . It could happen, that there are more connected components of  $Y$  at  $U$  when at  $X$ . In this case there will be more homomorphisms.

**Definition 1.2.2.** These homomorphisms are called Leray homomorphisms of the pair  $(X, Y)$  (or  $(U, Y)$ ). We denote the sum of Leray homomorphisms over all connected components of  $Y$  (or  $U$ ) at  $X$  by  $\phi_{(X,Y)}$  (resp.  $\phi_{(U,Y)}$ ).

**Definition 1.2.3.** We call a sequence  $X_1 < X_2 < \dots < X_n$  of strata of  $V$  a *chain*, if there is no such stratum  $Z$ , that  $X_i < Z < X_{i+1}$  for some  $i$ . We call  $n$  the *length* of the chain  $X_1 < X_2 < \dots < X_n$ .

**Theorem 1.2.2.** Consider such a pair  $X < Y$  of strata of  $V$  that all chains with the first stratum  $X$  and the last stratum  $Y$  have length 3. Let  $X < Z_1 < Y, X < Z_2 < Y, \dots, X < Z_p < Y$  be all such chains. Then

$$\phi_{(Z_1,Y)} \circ \phi_{(X,Z_1)} + \phi_{(Z_2,Y)} \circ \phi_{(X,Z_2)} + \dots + \phi_{(Z_p,Y)} \circ \phi_{(X,Z_p)} = 0.$$

*Proof:* Let  $a \in H_n(X)$  and  $\alpha : A \rightarrow X$  its representative. Let  $U$  be an open neighborhood of  $\alpha(A)$  in  $X$  compactly contained in  $X$ . Let  $N_{U,\delta}$  be as before. The set  $L_{U,\delta} = N_{U,\delta} \cap (Y \cup Z_1 \cup Z_2 \dots \cup Z_p)$  is a closed subset of  $N_{U,\delta}$  consisting of strata of  $N_{U,\delta}$ . Therefore,  $L_{U,\delta}$  is a stratified space with strata  $Y \cap N_{U,\delta}, Z_1 \cap N_{U,\delta}, \dots, Z_p \cap N_{U,\delta}$ .

For simplicity, we'll write  $N_U$  and  $L_U$  instead of  $N_{U,\delta}$  and  $L_{U,\delta}$ .

Let  $\beta_i : B_i \rightarrow Z_i \cap N_U, 1 \leq i \leq p$  be representatives of the  $\phi_{(X,Z_i)}(a)$  constructed as in the Theorem 1.2.1. Let  $U_i$  be open neighborhoods of  $\beta_i(B_i)$  in  $Z_i$  compactly contained in  $Z_i$ . Let  $N_{U_i,\delta_i}$  be as before.

For simplicity, we'll write  $N_{U_i}$  instead of  $N_{U_i,\delta_i}$ .

Hypersurfaces  $S_i = N_{U_i} \cap N_U \cap Y$  subdivide  $N_U \cap Y$  into  $p+1$  pieces:  $D_i = \{x \in N_U \cap Y \cap U_{Z_i} \mid \rho_{Z_i}(x) < \delta_i\}$  and all the rest  $D = (N_U \cap Y) - \bigcup D_i$ . Now  $D$  is a manifold with boundary  $\bigcup S_i$ . In other words, we cut from  $N_U \cap Y$  small neighborhoods of  $Z_i$ 's.

The map  $\pi_{X,Y}|_{S_i}$  is a submersion as a composition of two submersions:  $\pi_{Z_i,Y}|_{S_i}$  and  $\pi_{X,Z_i}|_{N_U \cap Z_i}$  (first map is a submersion to the  $N_U \cap Z_i$  as a restriction of the submersion  $\pi_{Z_i,Y}|_{N_{U_i} \cap Y}$  to the preimage of a hypersurface). By the Ehresmann's Lemma for manifolds with boundary, the map  $\pi_{X,Y}|_D$  is a locally trivial fibration, as well as its restrictions to the  $S_i$ 's.

A representative of the  $\phi_{(Z_i,Y)} \circ \phi_{(X,Z_i)}(a)$  can be constructed from  $\beta_i : B_i \rightarrow Z_i \cap N_U$  using the fibration  $\pi_{Z_i,Y}|_{N_{U_i}}$ . But since  $\beta_i(B_i) \subset N_U$ , we can restrict this fibration to the preimage of  $N_U \cap Z_i$ , which is, because of the commutation relations,  $S_i = N_{U_i} \cap N_U \cap Y$ .

We have the following diagram:

$$\begin{array}{ccc} S_i & \xleftarrow{\gamma_i} & C_i \\ \downarrow & & \downarrow \\ N_U \cap Z_i & \xleftarrow{\beta_i} & B_i \\ \downarrow & & \downarrow \\ U & \xleftarrow{\alpha} & A \end{array}$$

where  $\gamma_i : C_i \rightarrow S_i$  represents the class  $\phi_{(Z_i, Y)} \circ \phi_{(X, Z_i)}(a)$ . According to the commutation relations, the composition of two left vertical arrows on the diagram is the fibration  $\pi_{X, Y}|_{S_i}$ . Thus one can construct the representative  $\gamma_i : C_i \rightarrow S_i$  immediately from  $\alpha : A \rightarrow U$  using  $\pi_{X, Y}|_{S_i}$ .

One should choose the orientation of  $S_i$  carefully in order to get the right sign. Let  $\omega_X$  define the orientation of  $X$  (at some point in  $U$ ). Let  $\omega_{\pi_{Z_i, Y}}$  define the correct orientation of the fiber of  $\pi_{Z_i, Y}|_{S_i}$  (which is the same as the fiber of  $\pi_{Z_i, Y}|_{N_{U_i} \cap Y}$ ) and  $\omega_{\pi_{X, Z_i}}$  defines the correct orientation of the fiber of  $\pi_{X, Z_i}|_{N_U \cap Z_i}$ . (These forms are defined respectively on  $Y$  and  $Z_i$ . When we say that a form defines an orientation of a submanifold, we mean that its restriction to this submanifold defines the orientation.) That means that  $\pi_{X, Z_i}^*(\omega_X) \wedge \omega_{\pi_{X, Z_i}}$  defines the correct orientation of  $N_U \cap Z_i$  and  $\pi_{Z_i, Y}^*(\pi_{X, Z_i}^*(\omega_X) \wedge \omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}} = \pi_{X, Y}^*(\omega_X) \wedge \pi_{Z_i, Y}^*(\omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}}$  defines the correct orientation of  $S_i$ .

Let's check that it coincide with the natural orientation of the  $S_i$  as the boundary of  $D$ . That means that

$$-d\rho_{Z_i, Y} \wedge \pi_{X, Y}^*(\omega_X) \wedge \pi_{Z_i, Y}^*(\omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}}$$

should define the correct orientation of  $D$ , or

$$d\rho_{Z_i, Y} \wedge d\rho_{X, Y} \wedge \pi_{X, Y}^*(\omega_X) \wedge \pi_{Z_i, Y}^*(\omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}}$$

should define the correct orientation of  $Y$ . Enough to check that

$$d\rho_{X, Y} \wedge \pi_{X, Y}^*(\omega_X) \wedge \pi_{Z_i, Y}^*(\omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}}$$

defines the correct orientation of the  $N_{U_i} \cap Y$ , which is true, because

$$\begin{aligned} & d\rho_{X, Y} \wedge \pi_{X, Y}^*(\omega_X) \wedge \pi_{Z_i, Y}^*(\omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}} = \\ & = d\rho_{X, Y} \wedge \pi_{Z_i, Y}^*(\pi_{X, Z_i}^*(\omega_X) \wedge \omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}} = \\ & = \pi_{Z_i, Y}^*(d\rho_{X, Z_i} \wedge \pi_{X, Z_i}^*(\omega_X) \wedge \omega_{\pi_{X, Z_i}}) \wedge \omega_{\pi_{Z_i, Y}}, \end{aligned}$$

and  $d\rho_{X, Z_i} \wedge \pi_{X, Z_i}^*(\omega_X) \wedge \omega_{\pi_{X, Z_i}}$  defines the correct orientation of  $Z_i$ . On the last step we use that  $\pi^*$  commutes with wedge-product and  $\pi_{Z_i, Y}^*(d\rho_{X, Z_i}) = d\rho_{X, Y}$ .

Finally, we have the following situation: there is a fibration  $\pi_{X, Y}|_D : D \rightarrow U$ , where  $D$  is a manifold with boundary  $S = \bigcup S_i$ . A representative  $\gamma : C \rightarrow S \subset Y$  of the class  $\sum \phi_{(Z_i, Y)} \circ \phi_{(X, Z_i)}$  can be constructed using the restriction of this fibration to the boundary:  $\pi_{X, Y}|_S : S \rightarrow U$  (where  $S$  is oriented as boundary of  $D$ ). Then the chain, constructed in the same way, but using the whole fibration  $\pi_{X, Y}|_D : D \rightarrow U$  have the boundary exactly equal to our representative  $\gamma : C \rightarrow S$ .  $\square$

**Remark.** Let  $U \subset X$  be an open subset in  $X$ . One can construct such a small neighborhood  $W$  of  $U$  that for any stratum  $Y$  connected components of  $Y$  at  $U$



are in one-to-one correspondence with connected components of  $Y \cap W$  (Similarly as in the previous chapter).  $W$  is naturally stratified by connected components of the strata of the original space intersected with  $W$ . Applying Theorem 1.2.2 to  $W$  one can get relations, which don't follow from Theorem 1.2.2 applied to the original space.

### 1.3 Dual homomorphism and spectral sequence.

In this chapter the coefficient ring is always  $\mathbb{R}$ . For simplicity, we skip it in the notations.

**Theorem 1.3.1.** *Leray homomorphism  $\phi_{X,Y} : H_n(X) \rightarrow H_{n+l-m-1}(Y)$  ( $\dim X = m, \dim Y = l$ ) is up to a sign dual to the boundary homomorphism  $\partial_{Y,X} : H_{m-n+1}^{BM}(Y) \rightarrow H_{m-n}^{BM}(X)$ , where  $H^{BM}$  are Borel-Moore homologies (relative homologies of the one point compactification modulo infinity point).*

*More precisely,  $\phi_{X,Y}$  is dual to  $(-1)^{lm+(1-n)(l-m)}\partial_{Y,X}$ .*

*Proof:* By Poincare duality, the intersection form  $H_*(M) \times H_{d-*}^{BM}(M) \rightarrow \mathbb{R}$  is well defined and non-degenerate (where  $M$  is a smooth oriented manifold and  $\dim M = d$ ). Therefore, the only thing we need to check is that for any classes  $a \in H_n(X)$  and  $b \in H_{m-n+1}^{BM}(Y)$ ,

$$(a, \partial_{Y,X} b) = (-1)^{lm+(1-n)(l-m)} (\phi_{X,Y}(a), b).$$

Note, that

$$H_{m-n+1}^{BM}(Y) = H_{m-n+1}^{BM}(Y \cup X, X)$$

(that's how one construct the homomorphism  $\partial_{Y,X}$ ). By linearity it is enough to prove it for the classes representable by maps of simplicial complexes. Let  $\alpha : A \rightarrow X$  be a representative of  $a$ ,  $\alpha(A) \subset U \subset X$  and  $U$  open subset compactly contained in  $X$ . According to our construction,  $c = \phi_{X,Y}(a) \in H_{n+l-m-1}(Y)$  has a representative in  $W' = W \cap Y$ , where  $W$  is defined as in the Theorem 1.1.5. Therefore, it is enough to check the equality for  $b \in H_{m-n+1}^{BM}(W')$ .

By the Theorem 1.1.5,  $W'$  is diffeomorphic to the product  $(N_U \cap Y) \times (0, \Delta)$ . There is an isomorphism  $H_{m-n}^{BM}(N_U \cap Y) \rightarrow H_{m-n+1}^{BM}(W')$  constructed as follows — for a Borel-Moore cycle  $\kappa : K \rightarrow N_U \cap Y$  one gets the Borel-Moore cycle  $\beta : B \rightarrow W' = (N_U \cap Y) \times (0, \Delta)$ , where  $\beta = (\kappa, id)$  and  $B = K \times (0, \Delta)$ , with orientation given by  $-dt \wedge \omega_K$ , where  $\omega_K$  defines the orientation of  $K$  and  $t$  is the coordinate in  $(0, \Delta)$ . Therefore, it is enough to consider representatives of this type. Then the boundary  $\partial_{Y,X}(b)$  can be represented by  $\pi_{X,Y} \circ \kappa : K \rightarrow U$  with orientation given by  $\omega_K$ .

Suppose now that cycles  $\alpha : A \rightarrow U$  and  $\pi_{X,Y} \circ \kappa : K \rightarrow U$  intersect transversally. Lets construct a representative  $\gamma : C \rightarrow W'$  of  $\phi_{X,Y}(a)$  from the representative  $\alpha : A \rightarrow U$  as in Theorem 1.2.1. Consider a point  $y \in W'$ , such that the representatives  $\beta : B \rightarrow W'$  and  $\gamma : C \rightarrow W'$  both pass through  $y$ . Clearly,  $x = \pi_{X,Y}(y)$  belong

to the intersection of the  $\alpha : A \rightarrow U$  and  $\pi_{X,Y} \circ \kappa : K \rightarrow U$ . Furthermore, there is exactly one point of intersection of  $\beta : B \rightarrow W'$  and  $\gamma : C \rightarrow W'$  over each point  $x$  of the intersection of  $\alpha : A \rightarrow U$  and  $\pi_{X,Y} \circ \kappa : K \rightarrow U$ . Indeed,  $\gamma(C) \subset (N_U \cap Y)$  contain the whole fiber of  $\pi_{X,Y}|_{(N_U \cap Y)}$  over  $x$  and  $\beta(B) \cap W' = \kappa(K)$  — intersects this fiber exactly once, because  $K$  is mapped by  $\pi_{X,Y} \circ \kappa$  locally into at the preimage of  $x$  (since the intersection at  $x$  is transversal).

Lets check the signs. Since  $\alpha : A \rightarrow U$  and  $\pi_{X,Y} \circ \kappa : K \rightarrow U$  intersect transversally at  $x$ , there exist forms  $\omega_A$  and  $\omega_K$  on  $U$ , such that  $\alpha^*(\omega_A)$  defines positive orientation on  $A$  and  $(\pi_{X,Y} \circ \kappa)^*(\omega_K)$  defines positive orientation on  $K$ . Then  $s(\omega_A \wedge \omega_K)$  defines the orientation on  $U$ , where  $s$  is the sign of the intersection. Now the positive orientation of  $C$  is given by  $\gamma^*(\pi_{X,Y}^*(\omega_A) \wedge \omega_{\pi_{X,Y}})$ , where  $\omega_{\pi_{X,Y}}$  defines the orientation of the fiber of  $\pi_{X,Y}|_{(N_U \cap Y)}$ , and the positive orientation of the  $B$  is given by  $-dt \wedge \beta^*(\pi_{X,Y}^*(\omega_K)) = \beta^*(-d\rho_{X,Y} \wedge \pi_{X,Y}^*(\omega_K))$ . So, the orientation of  $W'$  is given by the form

$$-s'(\pi_{X,Y}^*(\omega_A) \wedge \omega_{\pi_{X,Y}} \wedge d\rho_{X,Y} \wedge \pi_{X,Y}^*(\omega_K)),$$

where  $s'$  is the sign of the intersection of  $\gamma : C \rightarrow W'$  and  $\beta : B \rightarrow W'$  at  $y$ .

On the other hand the orientation of  $W'$  is given by the form

$$d\rho_{X,Y} \wedge \pi_{X,Y}^*(s(\omega_A \wedge \omega_K)) \wedge \omega_{\pi_{X,Y}} = s(d\rho_{X,Y} \wedge \pi_{X,Y}^*(\omega_A) \wedge \pi_{X,Y}^*(\omega_K) \wedge \omega_{\pi_{X,Y}}).$$

If  $m = \dim X$ ,  $l = \dim Y$ ,  $n = \dim A$ , then  $\dim \omega_A = n$ ,  $\dim \omega_K = m - n$ ,  $\dim d\rho_{X,Y} = 1$ , and  $\dim \omega_{\pi_{X,Y}} = l - m - 1$ .

Thus,

$$-s' = (-1)^{(l-m-1)+n+(l-m-1)(m-n)} s,$$

or

$$\frac{s'}{s} = (-1)^{l-m+n+lm-ln-m^2+mn-m+n} = (-1)^{l-m+lm-ln+mn} = (-1)^{lm+(1-n)(l-m)}.$$

□

**Corollary.** The Leray homomorphisms  $\phi_{X,Y}$  don't depend on the choice of the control data at least modulo torsion.

Consider the following filtration of topological spaces:  $X \subset (X \cup Z_1 \cup \dots \cup Z_p) \subset (X \cup Z_1 \cup \dots \cup Z_p \cup Y)$ . One can consider the spectral sequence for the Borel-Moore homologies of this filtration. Note, that all three terms of the filtration are locally compact. Therefore, one can easily show that the first term of the spectral sequence is given by

$$\begin{aligned} E_{0,i}^1 &= H_i^{BM}(X), \\ E_{1,i}^1 &= H_{i+1}^{BM}(X \cup Z_1 \cup \dots \cup Z_p, X) = H_{i+1}^{BM}(Z_1) \oplus \dots \oplus H_{i+1}^{BM}(Z_p), \\ E_{2,i}^1 &= H_{i+2}^{BM}(X \cup Z_1 \cup \dots \cup Z_p \cup Y, X \cup Z_1 \cup \dots \cup Z_p) = H_{i+2}^{BM}(Y), \end{aligned}$$

$$\begin{aligned}\partial_{1,*}^1 &= \bigoplus_j \partial_{Z_j, X}, \\ \partial_{2,*}^1 &= \bigoplus_j \partial_{Y, Z_j}.\end{aligned}$$

$H_0^{BM}(Y)$	$H_1^{BM}(Y)$	$H_2^{BM}(Y)$	$H_3^{BM}(Y)$	$H_4^{BM}(Y)$	$\dots$
0	$\bigoplus H_0^{BM}(Z_i)$	$\bigoplus H_1^{BM}(Z_i)$	$\bigoplus H_2^{BM}(Z_i)$	$\bigoplus H_3^{BM}(Z_i)$	$\dots$
0	0	$H_0^{BM}(X)$	$H_1^{BM}(X)$	$H_2^{BM}(X)$	$\dots$

Now, the condition that the square of the boundary is zero for the first term of our spectral sequence is dual to the relation given by the Theorem 1.2.2. Easy to check, that the product of signs is independent of the dimension of the  $Z_i$  : if  $\dim Z_i = d$  then

$$(-1)^{dm+(1-n)(d-m)}(-1)^{ld+(1-(n+d-m-1))(l-d)} = (-1)^{-m+nm+nl}.$$

That gives another proof to the Theorem 1.2.2 modulo torsion.

## 2 Application to Parshin's Residues.

### 2.1 Point, Neighborhood and Local Coordinates.

Let  $V$  be an irreducible analytic variety of dimension  $n$ . Let  $x = V_0 \subset V_1 \subset \dots \subset V_n \subset V$  be a sequence of locally irreducible at  $x$  germs of subvarieties of dimension  $\dim V_k = k$ . After fixing such a sequence we can restrict everything to such a small neighborhood of  $x$  that all  $X_i$ 's have representatives in this neighborhood. So one can think of  $X_i$ 's as of subvarieties.

Consider the following diagram:

$$\begin{array}{c} V_n \supset V_{n-1} \supset \dots \supset V_1 \supset V_0 \\ \uparrow p_n \quad \uparrow p_n \\ \widetilde{V}_n \supset W_{n-1} \\ \uparrow p_{n-1} \\ \widetilde{W}_{n-1} \supset \dots \end{array}$$

$$\begin{array}{c} \dots \supset W_1 \\ \uparrow p_1 \\ \widetilde{W}_1 \supset W_0 \end{array}$$

where  $p_i$ 's are normalization maps and  $W_i$ 's are a full preimages of  $V_i$ 's.  $W_0$  is a zero dimensional variety, so it is just a finite set  $W_0 = a_1, \dots, a_N$ .

**Lemma 2.1.1.** *Each choice of a point  $a_\alpha \in W_0$  defines locally irreducible component of  $W_i$  at the image of  $a_\alpha$  for any  $0 \leq i \leq n$ .*

*Proof:* Let  $a_{\alpha,i}$  be the image of  $a_\alpha$  in  $\widetilde{W}_i$ .  $\widetilde{W}_i$  is locally irreducible at  $a_{\alpha,i}$ , since it is normal. Then the image of the germ of  $\widetilde{W}_i$  at  $a_{\alpha,i}$  defines an irreducible component of  $W_i$  at  $p_i(a_{\alpha,i})$ .  $\square$

**Definition 2.1.1.** The flag  $V_0 \subset V_1 \subset \dots \subset V_n \subset V$  together with a choice of a point  $a_\alpha \in W_0$  is called a *Parshin's point*.

**Notation.** Let's denote by  $W_{k,\alpha}$  the locally irreducible component of  $W_k$  at the image of  $a_\alpha$ .

$W_{i-1} \subset \widetilde{W}_i$  is a hypersurface in a normal variety. It follows that there exist a (meromorphic) function  $f_i$  on  $\widetilde{W}_i$  which has zero of order 1 at a generic point of  $W_{i-1}$ . Since meromorphic functions are the same on  $W_i$  and  $\widetilde{W}_i$ , one can consider  $f_i$  as a function on  $W_i$ . Then one can continue  $f_i$  to  $\widetilde{W}_{i+1}$  and so on. For simplicity, we denote all these functions by  $f_i$ . Now  $f_i$  is defined on  $V_n$ , and  $W_j$  and  $\widetilde{W}_j$  for  $j \geq i$ .

Consider a Whitney stratification  $V = \bigsqcup_{X \in \mathbf{S}} X$  such that all subvarieties  $V_i$  as well as divisors of zeroes and poles of  $f_i$ 's are unions of strata and all the strata are semianalytic subvarieties. Consider a control data  $\{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$  on this stratification.

**Lemma 2.1.2.** *Let  $p : \widetilde{M} \rightarrow M$  be a normalization map for an analytic variety  $M$  endowed with a structure of an abstract stratified space such that all strata are subanalytic. Then there is a unique (up to equivalence) structure of an abstract stratified space on  $\widetilde{M}$ , such that restrictions of  $p$  to strata of  $\widetilde{M}$  are coverings over strata of  $M$  and projections and tubular functions commute with  $p$  (In the sense that if  $X$  is a stratum of  $\widetilde{M}$  and  $p(X)$  is a stratum of  $M$ , then  $p \circ \pi_X = \pi_{p(X)} \circ p$  and  $p \circ \rho_X = \rho_{p(X)} \circ p$ ). Note that all strata of  $\widetilde{M}$  are again subanalytic.*

So now all the  $W_i$ 's and  $\widetilde{W}_i$  are stratified spaces. Note that divisors of  $f_i$ 's are unions of strata in all these stratifications (as far as  $f_i$  is defined).

**Notation.** Let  $X$  be an irreducible (complex analytic) variety considered with a fixed Whitney stratification. Then by  $X^0$  we denote the stratum of maximal dimension. If  $X$  is reducible, then by  $X^0$  we denote the union of strata of maximal dimension.

Pick a point  $a_\alpha$ . It is embedded into the normal curve  $\widetilde{W}_1$ . Consider a function  $f_1$  on  $\widetilde{W}_1$ . One can pick a small constant  $\epsilon_1$  and a small neighborhood  $U'_1$  of  $a_\alpha$  in  $\widetilde{W}_1^0 \cup a_\alpha$  such that  $f_1|_{U'_1} : U'_1 \rightarrow D = \{z \in \mathbb{C} : |z| < \epsilon_1\}$  is an isomorphism. On  $W_1$  our function  $f_1$  can be undefined at  $p_1(a_\alpha)$ . Thus we need to delete this point from the neighborhood. Let  $U_1 = p_1(U'_1 \setminus a_\alpha) \subset W_1^0$ . Then  $f_1|_{U_1} : U_1 \rightarrow A_1$  is an isomorphism, where  $A_1 = \{z \in \mathbb{C} : 0 < |z| < \epsilon_1\}$ .

$W_1$  is embedded into the normal surface  $\widetilde{W}_2$ . Moreover  $W_1^0$  is a smooth hypersurface in  $\text{reg}(\widetilde{W}_2)$ . Indeed, if there is a singular point of  $\widetilde{W}_2$  in  $W_1^0$ , then there should be another component of  $\{f_2 = 0\}$  passing through this point. That means that there is another stratum of our stratification containing this point in its closure. Which is impossible, because of the condition of the frontier. Therefore,  $\widetilde{W}_2^0 \cup U_1$  is a smooth manifold.

Each point  $x \in U_1$  has a small neighborhood  $U \subset \widetilde{W}_2^0 \cup U_1$  such that the pair  $(f_1, f_2)$  gives an isomorphism from  $U$  to a small ball in  $\mathbb{C}^2$ . We need the following lemma from theory of smooth manifolds:

**Lemma 2.1.3.** *Let  $Y \subset X$  be a smooth (real or complex) hypersurface in a manifold  $X$ . Let  $f$  be a smooth (real or complex) function on  $X$ , having zero of order 1 at each point of  $Y$ . Let  $F : Y \rightarrow \mathbb{F}^{n-1}$ , ( $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ) be a map, defining a diffeomorphism (real or complex) of  $Y$  to an open subset  $A \subset \mathbb{F}^{n-1}$ . Then there exist a smooth function  $\epsilon : A \rightarrow \mathbb{R}_+$  and a neighborhood  $U$  of  $Y$  in  $X$ , such that  $(f, F) : U \rightarrow \mathbb{F}^n$  defines a diffeomorphism of  $U$  to  $B = \{(x, y) \in \mathbb{F} \times \mathbb{F}^{n-1} : |x| < \epsilon(y)\}$ .*

It follows that there exist a (smooth) function  $\epsilon_2(z) : A_1 \rightarrow \mathbb{R}_+$  such that  $(f_1, f_2) : U'_2 \rightarrow A'_2$  is an isomorphism, where  $U'_2$  is an open neighborhood of  $U_1$  in  $\widetilde{W}_2^0 \cup U_1$  and  $A'_2 = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1| < \epsilon_1, |z_2| < \epsilon_2(z_1)\}$ . Similarly as before, functions  $f_1$  and  $f_2$  can be undefined on the image  $p_2(U_1)$ , so we need to delete it. Let  $U_2 = p_2(U'_2 \setminus U_1) \subset W_2^0$ . Then  $(f_1, f_2) : U_2 \rightarrow A_2$ , where  $A_2 = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1| < \epsilon_1, 0 < |z_2| < \epsilon_2(z_1)\}$ , is an isomorphism.

Proceeding in the same way, we get (smooth) functions  $\epsilon_i : A_{i-1} \rightarrow \mathbb{R}_+$ , where  $A_i = \{(z_1, \dots, z_i) : 0 < |z_1| < \epsilon_1, \dots, 0 < |z_i| < \epsilon_i(z_1, \dots, z_{i-1})\}$  and open sets  $U_i \subset W_i^0$ , such that  $(f_1, \dots, f_i) : U_i \rightarrow A_i$  are isomorphisms.

**Definition 2.1.2.**  $U_n$  is called an *open neighborhood* of the Parshin's point. Functions  $(f_1, \dots, f_n)$  are called local coordinates in this neighborhood.

## 2.2 Residues and Reciprocity Law.

Suppose now that we are given a meromorphic  $n$ -form  $\omega$  on  $V$ . One can assume that the divisor of zeros and poles of  $\omega$  is a union of strata. Thus our form is holomorphic on  $V_n^0 \supset U_n$ .

**Definition 2.2.1.** Let  $(\delta_1, \dots, \delta_n)$  be positive numbers such that  $\delta_i < \epsilon_i(\delta_1, \dots, \delta_{i-1})$ . By residue  $\text{res}_{\mathbf{V}, a_\alpha}(\omega)$ , where  $\mathbf{V} = \{V_n, \dots, V_0\}$ , we denote the integral  $\int_{\gamma_n} \omega$ , where  $\gamma_n = \{x \in U_n : |f_i(x)| = \delta_i, 0 < i \leq n\}$ .

**Remark.** Easy to see that this residue coincide with the residue defined by Parshin in [P1] and [P2]. Note, that one can use any Whitney stratification to construct the cycle  $\gamma_n$ , as far as all the elements of the flag  $\mathbf{V}$  are unions of strata (it is enough to shrink neighborhoods  $U_i$  in such a way that they fit into strata of the stratification).

Consider the images  $\widehat{U}_i := p_n \circ \dots \circ p_{i+1}(U_i) \subset V_i$ .

**Theorem 2.2.1.**  $\widehat{U}_i$  defines a connected component of  $V_i^0$  at  $\widehat{U}_{i-1}$ . Any loop  $\gamma : S^1 \rightarrow \widehat{U}_i$  can be deformed inside  $\widehat{U}_i$  to fit into any small neighborhood of  $\widehat{U}_{i-1}$ . The same statement holds for  $U_i$  instead of  $\widehat{U}_i$  and  $W_i^0$  instead of  $V_i^0$ .

The converse theorem is also true:

**Theorem 2.2.2.** *Suppose that  $\widehat{U}_i \subset V_i^0$  are open subsets, which represent connected components of  $V_i^0$  at  $\widehat{U}_{i-1}$ , and for all  $i$  any loop  $\gamma : S^1 \rightarrow \widehat{U}_i$  can be deformed inside  $\widehat{U}_i$  to fit into any small neighborhood of  $\widehat{U}_{i-1}$ . Then there is a unique Parshin's point defining the same connected components.*

**Definition 2.2.2.** We call a flag  $V_n \supset \cdots \supset V_0$  together with open subsets  $\widehat{U}_i$  an *extended flag*. We say that this extended flag is an extension of the flag  $V_n \supset \cdots \supset V_0$ . We say that two extensions are equivalent, if there exist a third extension of the same flag, such that  $\widehat{U}_i'' \subset \widehat{U}_i$  and  $\widehat{U}_i'' \subset \widehat{U}_i'$ , where  $\widehat{U}_i$  and  $\widehat{U}_i'$  are open subsets of the first two extensions and  $\widehat{U}_i''$  are open subsets of the third one.

Theorems 2.2.1 and 2.2.2 say that extended flags up to the equivalence and Parshin's points are in natural one-to-one correspondence.

**Remark.** Note that connected components of  $V^0$  are in one-to-one correspondence with irreducible components of  $V$ . The same holds locally.

**Theorem 2.2.3.** *The homology class of cycle  $\gamma_n$  in  $H_n(U_n)$  is exactly  $L_n \circ L_{n-1} \circ \cdots \circ L_1([V_0])$ , where  $L_i$  is the Leray homomorphism of the pair  $\widehat{U}_{n-1}$  and  $\widehat{U}_n$ .*

*Proof:* Consider cycles  $\gamma_k = \{x \in U_k : |f_i(x)| = \delta_i, 0 < i \leq k\}$  and cycles  $\widehat{\gamma}_k$ , which are preimages in  $W_{k,\alpha}$  of the representatives of  $L_k \circ \cdots \circ L_1([V_0])$ , constructed in the standard way. The choice of small enough  $\epsilon$ 's doesn't matter.

We prove by induction, that  $[\gamma_k] = [\widehat{\gamma}_k]$ , where  $[\gamma_k]$  and  $[\widehat{\gamma}_k]$  are homology classes of  $\gamma_k$  and  $\widehat{\gamma}_k$ . Since  $[\widehat{\gamma}_n] = L_n \circ \cdots \circ L_1([V_0])$ , this proves the theorem.

Lets first prove that  $[\gamma_1] = [\widehat{\gamma}_1]$  (one could avoid this by starting induction from  $\gamma_0 = \widehat{\gamma}_0 = a_{a_\alpha}$ , but we prefer to do the first step separately):

Consider cycle  $\gamma_1$ . This cycle can be viewed as a representative of the Leray homomorphism image of the zero dimensional cycle  $\gamma_0 = \{a_\alpha\}$  for the Leray homomorphism of the pair  $p_1(a_\alpha)$  and  $U_1$ . One should consider this pair as an abstract stratified space and take the following control data: the tubular neighborhood  $U_{p_1(a_\alpha)} = U_1$ , trivial projection  $\pi_{p_1(a_\alpha)} : U_1 \rightarrow p_1(a_\alpha)$  and tubular function  $\rho_{p_1(a_\alpha)} = |f_1|$ . Then our cycle is exactly the one obtained by the construction of Leray homomorphism with  $\epsilon = \delta_1$ .

One can consider another control data on the pair  $U_1, p_1(a_\alpha)$ . The control data on  $V$  induces control data on all the  $W_i$ 's. In particular, on the  $W_1$ . Restriction of this control data gives another control data on our pair. The cycle  $\widehat{\gamma}_1$  is exactly the one constructed using this control data. Since Leray homomorphism doesn't depend on the choice of control data, cycles  $\gamma_1$  and  $\widehat{\gamma}_1$  are equivalent.

The proof of the induction step is almost repeating of the above arguments.  $\square$

Let now  $V$  be an  $n$ -dimensional variety and  $\omega$  a meromorphic  $n$ -form on  $V$ . Let's fix a structure of an abstract stratified space of  $V$ , such that the divisor of zeros and poles of  $\omega$  is a union of strata and all strata are subanalytic.

**Theorem 2.2.4.** *Let  $V_n \supset \cdots \supset V_0$  be a sequence of irreducible subvarieties of  $V$ ,  $\dim V_i = i$ . Suppose that at least one of  $V_i$ 's is not a union of strata of our stratification. Then all the Parshin's residues corresponding to the sequence  $V_n \supset \cdots \supset V_0$  are zeros.*

**Corollary.** There is only finitely many non-zero Parshin's residues for a given meromorphic form.

Suppose now that we have a non-complete flag  $L_n \supset \cdots \supset L_{k+1} \supset L_{k-1} \supset \cdots \supset L_0$ , where  $0 < k < n$ . Let  $\omega$  be a meromorphic  $n$ -form on  $V$ . As always, consider a Whitney stratification of  $V$  with a control data, such that divisor of  $\omega$  and  $L_i$ 's are unions of strata. Let  $Q_i$ 's,  $0 \leq i \leq n$ ,  $i \neq k$  be open subsets in  $L_i^0$ , representing connected components of  $L_i^0$  at  $Q_{i-1}$  for  $i \neq k+1$  and let  $Q_{k+1}$  represent a connected component of  $L_{k+1}^0$  at  $Q_{k-1}$ . Let, similar to the Theorems 2.2.1 and 2.2.2, for any  $i$  any loop  $\gamma : S^1 \rightarrow Q_i$  can be deformed inside  $Q_i$  to fit into any small neighborhood of  $Q_{i-1}$  for  $i \neq k+1$ , and any small neighborhood of  $Q_{k-1}$  for  $i = k+1$ .

**Definition 2.2.3.** Non-complete flag  $L_n \supset \cdots \supset L_{k+1} \supset L_{k-1} \supset \cdots \supset L_0$  together with open subsets  $Q_i$  is called an *extended non-complete flag*.

**Definition 2.2.4.** We say that an extended flag defined by  $V_i$ 's and  $\widehat{U}_i$ 's is a continuation of an extended non-complete flag defined by  $L_i$ 's and  $Q_i$ 's, if  $V_i = Q_i$  for  $i \neq k$ , and there is an equivalent extended flag defined by  $V_i$ 's and  $\widehat{U}_i'$ 's, such that  $\widehat{U}_i' \subset Q_i$  for  $i \neq k$ .

The following Reciprocity Law easily follows from the Theorem 1.2.2:

**Theorem 2.2.5.** *Let's fix an extended non-complete flag. Consider all its continuations. For only finitely many of these continuations the Parshin's residue of  $\omega$  at the corresponding Parshin's point isn't zero. The sum of these finitely many non-zero residues is zero.*

**Corollary.** Summing up the relations over all extensions of a non-complete flag, one gets the Parshin's Reciprocity Law.

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